ELSEVIER

Topology and its Applications 68 (1996) 285–303

TOPOLOGY AND ITS APPLICATIONS

Reducing Dehn filling and toroidal Dehn filling

S. Boyer *, X. Zhang [1]

Département de Mathématiques, UQAM, P.O. Box 8888, Postal Station A, Montreal, Que., Canada H3C 3P8

Received 29 March 1995; revised 16 June 1995

Abstract

It is shown that if M is a compact, connected, orientable hyperbolic 3-manifold whose boundary is a torus, and r_1, r_2 are two slopes on ∂M whose associated fillings are respectively a reducible manifold and one containing an essential torus, then the distance between these slopes is bounded above by 4. Under additional hypotheses this bound is improved. Consequently the cabling conjecture is shown to hold for genus 1 knots in the 3-sphere.

Keywords: Dehn filling; Reducible slope; Essential torus slope; Cabling conjecture

AMS classification: 57M25; 57R65

0. Introduction

Let M be a connected, compact, orientable, irreducible 3-manifold such that ∂M is a torus. A *slope* on ∂M is a ∂M-isotopy class of essential, unoriented, simple, closed curves on ∂M, and the *distance* between two slopes r_1 and r_2, denoted $\Delta(r_1, r_2)$, is the minimal geometric intersection number amongst all curves representing the slopes. To each slope r on ∂M we associate the manifold $M(r)$ obtained by attaching a solid torus to M along ∂M in such a way that the meridianal slope of the solid torus is identified with r.

Now consider two distinct slopes r_1 and r_2 on ∂M. Recently, there has been much work done on the problem of determining how constraints on the topology of $M(r_1)$ and $M(r_2)$ put constraints on $\Delta(r_1, r_2)$. For instance Gordon and Luecke [5] have shown that if $M(r_1)$ and $M(r_2)$ are reducible manifolds, then $\Delta(r_1, r_2) = 1$ and Gordon [4] has shown that if M is a hyperbolic manifold such that $M(r_1)$ and $M(r_2)$ are manifolds each of which contains an essential torus, then $\Delta(r_1, r_2) \leqslant 5$ except for four specific

* Corresponding author. Partially supported by grants: NSERC OGP 0009446 and FCAR EQ 3518.

[1] Supported by a postdoctoral fellowship from the Centre de Recherches Mathematiques.

0166-8641/96/$15.00
SSDI 0166-8641(95)00061-5

manifolds M for which $\Delta(r_1, r_2) = 6, 7$ or 8 is possible. Earlier, Gordon and Litherland [8, Proposition 6.1] showed that when M is hyperbolic and $M(r_1)$ is a reducible manifold while $M(r_2)$ contains an incompressible torus, then $\Delta(r_1, r_2) \leqslant 5$ (see also [4, Theorem 1.2]). In this paper we improve this last result and show that 4 is an upper bound.

Theorem 0.1. *Let M be a compact, orientable, hyperbolic 3-manifold with ∂M a torus. If r_1 and r_2 are slopes on ∂M such that $M(r_1)$ is a reducible manifold while $M(r_2)$ contains an incompressible torus, then $\Delta(r_1, r_2) \leqslant 4$.*

It is unknown whether or not the bound 4 is optimal (we expect that it is not), though the following example shows that the distance 3 between a reducible slope and an essential torus slope may be realized.

Let W be the complement of the Whitehead link [11, Example 10, p. 68] and parameterize the slopes on each component of ∂W by the standard meridian-longitude coordinates. If M is the manifold obtained by Dehn filling one component of ∂W with slope 6, then it may be verified that

(i) M is a hyperbolic manifold,

(ii) $M(4)$ contains an incompressible torus and $M(1)$ is a reducible manifold.

Fact (i) can be verified using Jeff Weeks' SNAPPEA programme and fact (ii) can be verified through the use of the Montesinos trick.

Theorem 0.1 can be sharpened if we consider certain additional hypotheses. For instance, if $M(r_2)$ contains an incompressible torus T such that $T \cap \partial M$ is a 1-sphere, then $\Delta(r_1, r_2) \leqslant 1$ (Lemma 4.1). We note that this result is sharp by taking M to be as in the last paragraph and observing that $M(0)$ contains an incompressible torus which intersects ∂M in a circle.

Recall that the cabling conjecture asserts that no noncabled knot in S^3 admits a surgery yielding a reducible manifold. As a consequence of Lemma 4.1, we derive the following result.

Theorem 0.2. *Genus one knots in S^3 satisfy the cabling conjecture.*

The conjecture has been proved for several classes of knots in S^3 including satellite knots [12], strongly invertible knots [2], alternating knots [10], most knots with symmetry, including all Montesinos knots [9]. It is also known that 0-surgery on any nontrivial knot in S^3 yields an irreducible manifold [3] and that if some surgery on a nontrivial knot in S^3 yields a reducible manifold, then the surgery slope is an integer [6] and the resultant manifold contains a nontrivial lens space as a connected summand [7].

Theorem 0.1 will be proved by applying the combinatorial techniques developed in [1,4,8,13].

The authors would like to thank Paul Libbrecht for preparing the figures that appear in this paper.

1. Notations and definitions

All manifolds in this paper are understood to be orientable. We always assume that proper submanifolds meet in general position. For a manifold W, we use $\mathrm{int}(W)$ to denote its interior and ∂W to denote its boundary. A regular neighborhood in W of a subcomplex $Q \subset W$ will be denoted by $N(Q)$. By a surface we shall mean a compact, connected 2-manifold. A surface in a 3-manifold W is called *essential* if it is properly embedded and is either (i) incompressible, not parallel to a subsurface of ∂W, and not a 2-sphere, or (ii) a 2-sphere that does not bound a 3-ball in W. We note that if ∂M is a torus, then any essential surface in M is also ∂-incompressible. A 3-manifold is called *irreducible* if it does not contain an essential 2-sphere; otherwise it is called *reducible*. A knot K in a 3-manifold W is said to be a *cabled* knot if there is another knot K' in W such that $K \subset \partial N(K')$ and the winding number of K in the solid torus $N(K')$ is larger than one. A compact 3-manifold is said to be *cabled* if it is the exterior of a cabled knot. Note that a cabled 3-manifold either contains an essential torus or is a Seifert fibred space (thus is not a hyperbolic manifold).

In what follows, we shall assume all of the conditions listed in Theorem 0.1. We may further assume that $M(r_2)$ is irreducible as otherwise the main result of [8] cited in Section 0 implies that $\Delta(r_1, r_2) \leqslant 1$.

Let V_i be the solid torus attached to M in forming $M(r_i)$, $i = 1, 2$. Consider the family of essential 2-spheres in $M(r_1)$ which intersect V_1 in a family of meridianal discs, and let $S \subset M(r_1)$ be such a 2-sphere chosen so that $S \cap V_1$ has the minimal number, say n_1, of components. Similarly, let $T \subset M(r_2)$ be an essential torus which intersects V_2 in a collection of meridian discs, the number of which, say n_2, is minimal amongst all such tori. Note that as M is hyperbolic, $n_i > 0$, $i = 1, 2$.

Now if $F_1 = M \cap S$ and $F_2 = M \cap T$, then F_1 is an essential planar surface in M while F_2 is an essential punctured torus. We may assume that the number of components of $F_1 \cap F_2$ is minimal amongst all the surfaces in M isotopic to F_1 and transverse to F_2. Then no circle component of $F_1 \cap F_2$ bounds a disk in F_1 or F_2, and no arc component of $F_1 \cap F_2$ is boundary parallel in F_1 or F_2. Let Γ_1 (respectively Γ_2) be the graph in S (respectively in T) obtained by taking the arc components of $F_1 \cap F_2$ as edges and taking $S \cap V_1 = S - \mathrm{int}(F_1)$ (respectively $T \cap V_2 = T - \mathrm{int}(F_2)$) as fat vertices. Note that if x_j is a vertex of Γ_α, then ∂x_j is a boundary component of F_α.

We shall use the indices α and β to denote 1 or 2, with the convention that, when they are used together, $\{\alpha, \beta\} = \{1, 2\}$.

Number the vertices of Γ_α $x_1, \ldots, x_{n_\alpha}$ so that the corresponding components of ∂F_α appear consecutively on ∂M. By construction, each component ∂x_j of ∂F_α intersects each component ∂y_k of ∂F_β in exactly $\Delta = \Delta(r_1, r_2)$ points. The ends of the edges in Γ_α may be labeled by an integer $k \in \{1, 2, \ldots, n_\beta\}$ as follows. Let $*$ be the intersection of an edge e of Γ_α with one of its vertices, say x_j. Then $*$ is labeled k where y_k is the unique vertex of Γ_β such that $* = e \cap \partial x_j \cap \partial y_k$. Thus we may travel around ∂x_j so that the labels appear in the order $1, \ldots, n_\beta, \ldots, 1, \ldots, n_\beta$ (repeated Δ times).

We fix an orientation on F_α and let each component ∂x_j of ∂F_α have the induced orientation. Each component ∂x_j can be assigned a "+" or "−" sign depending on whether or not its orientation is parallel on ∂M to that on ∂x_1. Two vertices of Γ_α are called *parallel* if the corresponding boundary components of F_α have the same sign, and otherwise they are called *antiparallel*. Since F_1, F_2 and M are orientable, one has the following constraint on Γ_1 and Γ_2.

Parity rule 1.1. An edge connects parallel vertices of Γ_α if and only if it connects antiparallel vertices in Γ_β.

Remark 1.2. If S (respectively T) is separating in $M(r_1)$ (respectively in $M(r_2)$), then the number of vertices in Γ_1 (respectively in Γ_2) with a "+" sign is equal to the number of such vertices with a "−" sign. In particular, n_1 (respectively n_2) is an even integer.

Two edges are said to be parallel in Γ_α if they, together with some arcs in ∂F_α, bound a disk in F_α. A *cycle* σ in Γ_α is any subgraph which becomes homeomorphic to a circle, after the (fat) vertices of Γ_α have been shrunk to points. The *length* of a cycle is the number of edges which it contains. A *loop* is a length one cycle, and we will call a loop *trivial* if it bounds a disk face of the graph. Note that by construction, Γ_α has no trivial loops. We shall consider two parallel loops as a length two cycle. We call a cycle σ in Γ_2 *essential* if σ does not bound a disk in T. A cycle σ in Γ_α is called a *Scharlemann cycle* if it bounds a disk face of Γ_α disjoint from F_β and if the edges of σ connect parallel vertices of Γ_α and have the same two labels at their ends. Note that the two labels of a Scharlemann cycle in Γ_α are successive (mod n_β). A length two Scharlemann cycle will be called an S-cycle. Note that the two edges of an S-cycle are adjacent parallel edges connecting two parallel (perhaps equal) vertices. The disk face bounded by an S-cycle is referred to as its *S-disk*. A length two cycle $\sigma' = \{e_1', e_2'\}$ in Γ_α is called an *extended* S-cycle if there is an S-cycle $\sigma = \{e_1, e_2\}$ in Γ_α such that e_i and e_i' are parallel adjacent edges in Γ_α for $i = 1, 2$. The disk in F_α bounded by an extended S-cycle whose interior intersects the S-cycle is called the *extended* S-disk of the extended S-cycle.

The *reduced* graph $\overline{\Gamma}_\alpha$ is the graph obtained from Γ_α by amalgamating each complete set of mutually parallel edges of Γ_α to a single edge.

2. Preparatory lemmas

Lemma 2.1. *Let W be a compact irreducible 3-manifold. Then any nonseparating torus T in W is essential in W.*

Proof. Let T' be a nonseparating torus in W. Any compression of T' which surgered it along an essential curve would produce a nonseparating 2-sphere, contradicting the irreducibility of W. Thus T' must be incompressible. As it clearly cannot be ∂-parallel, T' is essential in W. □

Lemma 2.2. (1) *If T is nonseparating in $M(r_2)$, then Γ_1 does not contain an S-cycle.*

(2) *If S is nonseparating in $M(r_1)$, then Γ_2 does not contain an S-cycle.*

Proof. (1) Suppose that $\{e_1, e_2\}$ is an S-cycle in Γ_1 with label pair $\{r, r+1\}$. Let D be the S-disk of the given S-cycle and let H be the part of the attached solid torus V_2 which lies between the disks x_r and x_{r+1} and is disjoint from other vertices of Γ_2. Then $Q = (\partial H \cup T) - \text{int}(x_r \cup x_{r+1})$ is a nonseparating closed genus two surface in $M(r_2)$, $\text{int}(D) \cap Q$ is empty and $\partial D \subset Q$ is a nonseparating simple closed curve in Q. Compressing Q with D, we obtain a new nonseparating torus T' in $M(r_2)$, which is essential by Lemma 2.1. But the intersection of T' with V_2 has two fewer components than does T, which is impossible by the minimality of n_2. Thus Γ_1 cannot contain an S-cycle.

(2) The proof of part (2) is similar to that of part (1). □

Lemma 2.3. $n_1 \geqslant 3$.

Proof. First observe that if n_1 is either 1 or 2, F_1 would be either a 2-disk or an annulus. In the former case, M would have a compressible boundary, and thus it would be a solid torus. In the latter case, M would either admit an essential torus or be Seifert fibred. In any event, none of the above possibilities can arise owing to the fact that M is a hyperbolic manifold. □

Lemma 2.4 [1, Corollary 2.6.7]. *If Γ_α has more than $n_\beta/2$ mutually parallel edges connecting parallel vertices of Γ_α, then two of these edges form an S-cycle of Γ_α.*

Lemma 2.5. (1) *Γ_2 cannot contain two S-cycles with different label pairs.*

(2) *Γ_2 cannot contain an extended S-cycle.*

(3) *Γ_2 cannot have more than $n_1/2 + 1$ mutually parallel edges connecting parallel vertices.*

(4) *Suppose that e_1, e_2 are two parallel edges of Γ_2 connecting parallel vertices of Γ_2. If they have a label r in common, then they form an S-cycle.*

Proof. According to Lemma 2.3, $n_1 \geqslant 3$. In the case that $n_1 = 3$, Remark 1.2 implies that S is a nonseparating 2-sphere in $M(r_1)$, and thus Lemma 2.2(2) shows that Γ_2 cannot have an S-cycle. In particular, parts (1) and (2) of the lemma do not arise. Similarly, applying Lemma 2.4 (respectively Parity rule 1.1), we see that part (3) (respectively part (4)) does not occur. Thus we may assume that $n_1 \geqslant 4$. But then the proof of Lemma 2.5 proceeds exactly as in the proofs of [13, Lemmas 2.2–2.4]. □

The following lemma follows from the argument of [8, Proposition 1.3].

Lemma 2.6. *Γ_2 cannot have n_1 mutually parallel edges, as otherwise M would be cabled, and thus could not be hyperbolic.*

Lemma 2.7. *Let W be an irreducible 3-manifold which contains an essential torus T. Suppose that T' is a torus in W such that $T \cap T'$ is an annulus which is essential in both T and T'. Then if T' compresses in W, it bounds a solid torus U in W such that $T \cap \operatorname{int}(U) = \emptyset$.*

Proof. Using the irreducibility of W, we see that if T' compresses in W, then either it bounds a solid torus in W or it is contained in a 3-ball in W. The latter can never occur, as otherwise the annulus $T \cap T'$ would be homotopically trivial in W, and thus also in T, contradicting our hypotheses. It follows that there is a solid torus U in W whose boundary is T'. Now W cannot be a solid torus, and so in particular, T' separates W, and thus $T \subset U$ or $T \subset (W - \operatorname{int}(U))$. The incompressibility of T in W implies that the latter must occur, in other words $T \cap \operatorname{int}(U) = \emptyset$. This completes the proof of the lemma. □

Lemma 2.8. *Suppose that $\{e_1, e_2\}$ is an S-cycle in Γ_1 with label pair $\{r, r+1\}$. Then $e_1 \cup e_2 \cup x_r \cup x_{r+1}$ is an essential cycle in Γ_2.*

Proof. Suppose, otherwise, that $e_1 \cup e_2 \cup x_r \cup x_{r+1}$ is contained in a disk B of T. Let D be the S-disk of the S-cycle $\{e_1, e_2\}$ in Γ_1, and let H be the portion of the attached solid torus V_2 which lies between x_r and x_{r+1} and is disjoint from the other vertices of Γ_2. Then a regular neighborhood of $B \cup H \cup D$ in $M(r_2)$ is a punctured nontrivial lens space, which implies that $M(r_2)$, being irreducible, is itself a lens space. But then $M(r_2)$ cannot contain an essential torus, contradicting the defining property of r_2. Thus the cycle $e_1 \cup e_2 \cup x_r \cup x_{r+1}$ is essential in T. □

In the case that T separates $M(r_2)$ into two submanifolds X_1 and X_2, we shall say that an S-cycle *lies on the X_i-side* of T if its associated S-disk lies in X_i.

Lemma 2.9. *If $n_2 \geqslant 3$ and Γ_1 contains an S-cycle, then T separates $M(r_2)$ and the side containing the S-disk admits the structure of a Seifert fibred space with base orbifold the 2-disk having exactly two cone points. Furthermore, if Γ_1 contains two S-cycles with disjoint label pairs, then they both lie to the same side of T.*

Proof. Denote the S-cycle by $\{e_1, e_2\}$ and its labels by $\{r, r+1\}$. In Γ_2, e_1, e_2 connect the vertices x_r and x_{r+1}, which are antiparallel, and so unequal, by the Parity rule 1.1. Let D be the S-disk and let H be the portion of the attached solid torus V_2 which lies between x_r and x_{r+1} and is disjoint from the other vertices of Γ_2.

According to Lemma 2.2(1), the existence of the S-cycle implies that T separates $M(r_2)$ into two irreducible submanifolds X_1 and X_2. Then from Remark 1.2 and our hypothesis, we may assume that $n_2 \geqslant 4$.

Suppose now that the disk D lies in X_1. Then $H \subset X_1$ also. Let A_1 be a thin regular neighbourhood in T of the essential cycle $e_1 \cup e_2 \cup x_r \cup x_{r+1}$ (Lemma 2.8) and let U_1 be a regular neighborhood in X_1 of $A_1 \cup H \cup D$. Then ∂U_1 is a torus which intersects V_2 in $2 < n_2$ meridian discs, which shows that ∂U_1 must be compressible in $M(r_2)$. Since

$T = \partial X_1$ is essential in $M(r_2)$, ∂U_1 is also compressible in X_1. Further note that ∂U_1 intersects T in an annulus which is essential in both T and ∂U_1. Thus by Lemma 2.7, U_1 is a solid torus.

Next let $U_2 = X_1 - \mathrm{int}(U_1)$. The boundary of U_2 is a torus which intersects V_2 in $n_2 - 2$ meridian disks, and so as before, we may deduce that U_2 is a solid torus. Now X_1 is the manifold obtained by gluing U_1 and U_2 along the annulus $A_2 = \partial U_1 - \mathrm{int}(A_1)$, which is essential in both U_1 and U_2. We may therefore construct a Seifert structure on X_1 whose base orbifold is a 2-disk with at most two cone points. As $T = \partial X_1$ is incompressible in X_1, there are exactly two cone points. Finally observe that $e_1 \cup e_2 \cup x_r \cup x_{r+1}$ is isotopic to a fibre in this structure.

Assume now that Γ_1 contains another S-cycle $\{e_1', e_2'\}$ with label pair $\{s, s+1\}$ disjoint from $\{r, r+1\}$. Let D' be the S-disk and let H' be the portion of the attached solid torus V_2 which lies between x_s and x_{s+1} and is disjoint from the other vertices of Γ_2.

Suppose that $D' \subset X_2$ and construct A_1', A_2', U_1' and U_2' as in the previous paragraph. Then X_2 admits the structure of a Seifert fibred space having base orbifold the 2-disk with exactly two cone points. Furthermore, $e_1' \cup e_2' \cup x_s \cup x_{s+1}$ is isotopic to a fibre in this structure. But according to Lemma 2.8, $e_1' \cup e_2' \cup x_s \cup x_{s+1}$ is an essential cycle in T, and our hypotheses imply that it is disjoint from the essential cycle $e_1 \cup e_2 \cup x_r \cup x_{r+1}$. Thus these two cycles are parallel on T, and so $M(r_2) = X_1 \cup X_2$ admits the structure of a Seifert fibred space over the 2-sphere with exactly four singular fibres and for which A_1 is a vertical annulus. Now we may assume that $U_1 \cap U_1' = \emptyset$ and so both $U_1 \cup U_2'$ and $\overline{M(r_2) - (U_1 \cup U_2')} = U_1' \cup U_2$ are Seifert fibred over the 2-disk with exactly two singular fibres each. Hence $\partial(U_1 \cup U_2')$ is an incompressible torus in $M(r_2)$. But this torus intersects V_2 in at most $n_2 - 2$ meridianal disks, contradicting our choice of n_2. Thus D' cannot lie in X_2, that is $D' \subset X_1$. $\square$

A similar analysis to that used in the previous lemma proves the following one.

Lemma 2.10. *If $n_2 \geqslant 3$, then Γ_1 cannot contain an extended S-cycle.*

Proof. Suppose, otherwise, that $\{e_1', e_2'\}$ is an extended S-cycle in Γ_1 which extends the S-cycle $\{e_1, e_2\}$. By Lemma 2.2(1), T is separating, and therefore by Remark 1.2 and our hypothesis, we may assume that $n_2 \geqslant 4$.

Denote by $\{r, r+1\}$ the labels of the S-cycle, so that the extended S-cycle has labels $\{r-1, r+2\}$ (mod n_2). Then in Γ_2, e_1', e_2' connect the vertices x_{r-1} and x_{r+2}, which are antiparallel by the Parity rule 1.1. Further note that as $n_2 \geqslant 4$, $\{x_{r-1}, x_{r+2}\}$ and $\{x_r, x_{r+1}\}$ are disjoint in Γ_2. Let D be the extended S-disk and let H be the portion of the attached solid torus V_2 which lies between x_{r-1} and x_{r+2} and contains x_r and x_{r+1}.

Claim. *$e_1' \cup e_2' \cup x_{r-1} \cup x_{r+2}$ is an essential cycle of Γ_2.*

Proof. If $e_1' \cup e_2' \cup x_{r-1} \cup x_{r+2}$ is contained in a disk in T, we may use Lemma 2.8 to find such a disk, B say, which is disjoint from $e_1 \cup e_2 \cup x_r \cup x_{r+1}$. Then it is easy to

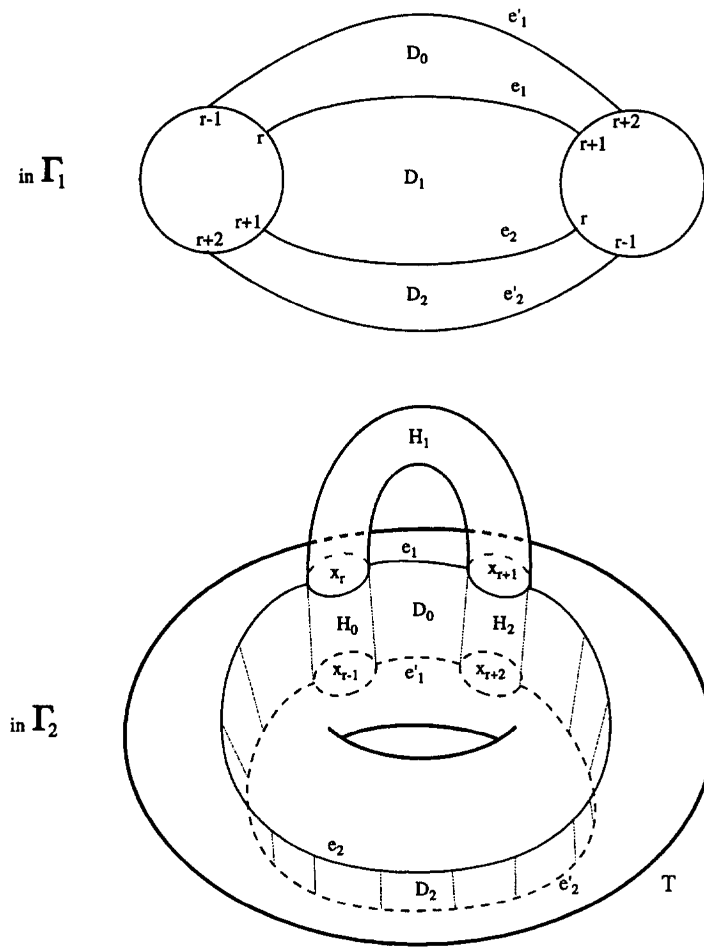

Fig. 1.

see that a regular neighborhood of $B \cup H \cup D$ in $M(r_2)$ is a nontrivial punctured lens space, which leads to a contradiction as in the proof of Lemma 2.8.

Thus $e_1 \cup e_2 \cup x_r \cup x_{r+1}$ and $e_1' \cup e_2' \cup x_{r-1} \cup x_{r+2}$ are disjoint, essential cycles of Γ_2, and so are parallel on T. Our goal is to use this observation along with Lemma 2.7 to construct a new essential torus in $M(r_2)$ which intersects V_2 in fewer meridian disks than does T, contradicting our initial choices.

Now the edges e_1 and e_2 divide the extended S-disk D into three subdisks, denoted by D_0, D_1, D_2, with D_1, say, being the S-disk. The vertices x_r and x_{r+1} divide H into three parts, denoted by H_0, H_1, H_2, with H_1, say, being that part which lies between x_r and x_{r+1} (Fig. 1). By Lemma 2.2(1), T is a separating torus in $M(r_2)$. Let X_1 and X_2 be the two (irreducible) submanifolds in $M(r_2)$ bounded by T. We may assume that X_1

contains H_0, H_2, D_0, D_2 and that X_2 contains H_1, D_1. Letting the topological closure operation be denoted by an overbar, we have

(i) $\overline{T - \{x_{r-1}, x_r, x_{r+1}, x_{r+2}\}}$ splits into two annuli A_1, A_2, which are joined along the four arcs in $\partial D_0 \cap T$ and $\partial D_2 \cap T$.

(ii) For $j = 0, 2$, $M \cap \partial H_j$ is divided into two disks which intersect along the two arcs $\partial D_0 \cap \partial H_j$ and $\partial D_2 \cap \partial H_j$. Further, H_j provides an isotopy of one disk to the other, relative to these two arcs.

There is a unique disk in $M \cap \partial H_0$ as described in (ii) which shares two boundary arcs with the annulus A_1, and similarly for $M \cap \partial H_2$. The union of these two disks with D_0 and D_2 forms a band B_1 with two boundary components, i.e., an annulus, which is properly embedded in X_1 and for which $\partial B_1 = \partial A_1$ (see Fig. 1). Now $T_1 = A_1 \cup B_1$ is a torus in X_1 which, after being pushed off of $H_0 \cup H_2$, will intersect V_2 in a collection of fewer than n_2 meridian discs. Thus by the choice of n_2, T_1 compresses in $M(r_2)$, and hence also in X_1. Since $T \cap T_1 = A_1$ is an essential annulus in both T and T_1, we may invoke Lemma 2.7 to conclude that T_1 bounds a solid torus U_1 in X_1. Thus X_1 decomposes into two pieces $X_1 = U_1 \cup_{B_1} \overline{X_1 - U_1}$. In fact $\overline{X_1 - U_1}$ is also a solid torus. To see this, consider the torus $T_2 = \overline{(T - A_1)} \cup B_1 \subset X_1$ and note that T_2 may be pushed through $H_0 \cup H_2$ by using the isotopies described in (ii). It follows that T_2 is isotopic to a torus which intersects V_2 in a collection of fewer than n_2 meridian discs. Thus T_2 is compressible in $M(r_2)$, and as above, it bounds a solid torus U_2 in X_1, which is necessarily $\overline{X_1 - U_1}$. We conclude that X_1 is the union of two solid tori U_1 and U_2, joined along the annulus A_1, which is essential in the boundary of each of them. X_1 therefore admits a Seifert structure whose base orbifold is a 2-disk which has at most two cone points. As $T = \partial X_1$ is incompressible in X_1, there are exactly two cone points. Note also that in this structure, A_1 is a union of fibres, that is A_1 is vertical.

Since $n_2 \geqslant 3$ and the given S-cycle lies to the X_2-side of T, Lemma 2.9 implies that there is also a Seifert structure on X_2 whose base orbifold is a 2-disk with exactly two cone points. More precisely, let A_3 be a thin regular neighbourhood in T of the cycle $e_1 \cup e_2 \cup x_r \cup x_{r+1}$ and let U_3 be a regular neighborhood in X_2 of $A_3 \cup H_1 \cup D_1$. Then the proof of Lemma 2.9 shows that both U_3 and $U_4 = X_2 - \operatorname{int}(U_3)$ are solid tori which intersect along an annulus which is essential in both U_3 and U_4 and vertical with respect to the Seifert structure on X_2. By construction, both A_1 and A_3 are vertical annuli in this structure.

Next we consider $M(r_2)$. Now $A_1 \subset T$ is a vertical annulus in both X_1 and X_2, and so the two fibrings are isotopic when restricted to T. Hence $M(r_2)$ is Seifert fibred over the 2-sphere with exactly four singular fibres in such a way that $A_3 = U_2 \cap U_3$ is vertical. Then $U_2 \cup U_3$ and $\overline{M(r_2) - (U_2 \cup U_3)}$ are Seifert fibred over the 2-disk with exactly two singular fibres each. Hence $\partial(U_2 \cup U_3)$ is an incompressible torus in $M(r_2)$. But this torus intersects V_2 in at most $n_2 - 2$ meridianal disks, contradicting our choice of n_2. Thus Γ_1 can contain no extended S-cycle. $\square$

Lemma 2.11. *If $n_2 \geqslant 3$, then Γ_1 cannot have more than $n_2/2 + 2$ mutually parallel edges connecting parallel vertices of Γ_1.*

Proof. Otherwise using Lemma 2.4, Γ_1 would contain an S-cycle, and so T would be separating by Lemma 2.1(1). It follows from Remark 1.2 that n_2 is an even integer. It is now easy to see that Γ_1 contains an extended S-cycle, contradicting Lemma 2.10. □

3. Proof of Theorem 0.1 when $n_2 \geqslant 3$

In this section we prove Theorem 0.1 under the additional hypothesis that $n_2 \geqslant 3$. The case that $n_2 \leqslant 2$ will be dealt with separately in Section 4. To obtain a contradiction, we shall suppose that $\Delta = \Delta(r_1, r_2) \geqslant 5$. Recall that $\overline{\Gamma}_2$ denotes the reduced graph of Γ_2 in T.

Claim 3.1. *Each vertex of $\overline{\Gamma}_2$ has valency at least* 6.

Proof. Suppose otherwise that $\overline{\Gamma}_2$ has a vertex of valency at most 5. Then since $\Delta \geqslant 5$, in Γ_2 there is a family of at least n_1 mutually parallel edges. But this contradicts Lemma 2.6. □

Claim 3.2. *Each vertex of $\overline{\Gamma}_2$ has valency* 6.

Proof. A *face* of Γ_2 is a component of the complement of an open regular neighbourhood of Γ_2 in T. If v and e denote the number of vertices and edges in Γ_2, then it is easy to see that

$$0 = \chi(T) = v - e + \sum \chi(F),$$

where the sum is taken over the faces F of Γ_2. By Claim 3.1, each vertex of $\overline{\Gamma}_2$ has valency at least 6. Hence we have $2e \geqslant 6v$, that is $v \leqslant e/3$. Now let d be the number of disk faces of $\overline{\Gamma}_2$. By construction, each disk face of $\overline{\Gamma}_2$ has at least three sides. Hence we have $2e \geqslant 3d$ and thus $d \leqslant 2e/3$. Substituting these inequalities into the identity above shows that

$$0 = v - e + \sum \chi(F) \leqslant v - e + d \leqslant \frac{e}{3} - e + \frac{2e}{3} = 0.$$

It follows that all of the above inequalities are in fact equalities. In particular the claim must hold. □

Thus the ends of the edges incident to a given vertex of Γ_2 can be partitioned into six families such that each family consists of ends of mutually parallel edges of Γ_2.

Claim 3.3. *Among the six families of ends around each vertex y of Γ_2, there is at most one family which are ends of edges connecting y to a parallel vertex.*

Proof. Suppose otherwise that there are two such families. By Lemma 2.5(3), two such families occupy at most $2(n_1/2 + 1) = n_1 + 2$ ends. By Lemma 2.6, the remaining four families occupy at most $4(n_1 - 1)$ ends. Hence there is a total of at most $5n_1 - 2$ ends of edges incident to y. But we have assumed that there are $\Delta n_1 \geqslant 5n_1$ ends of edges of Γ_2 incident to y and so a contradiction is derived. □

Combining Claim 3.3 and Lemma 2.5 we deduce that at each vertex y of Γ_2, the ends of edges which belong to edges of Γ_2 connecting y to antiparallel vertices are successive, and their number is at least $\Delta n_1 - (n_1/2 + 1)$. Therefore by the parity rule, we have

Claim 3.4. *Let $r \in \{1, \ldots, n_2\}$ be any given label. Among the Δ ends of edges with the label r around each vertex x_j of Γ_1, there is at most one which belongs to an edge connecting x_j to an antiparallel vertex of Γ_1.*

Now we consider the reduced graph $\overline{\Gamma}_1$ of Γ_1 in S.

Claim 3.5. *$\overline{\Gamma}_1$ has a vertex of valency at most 5.*

Proof. Suppose otherwise that every vertex of $\overline{\Gamma}_1$ has valency at least 6. Let v, e, d be the number of vertices, edges and disk faces of $\overline{\Gamma}_1$. Then arguing as in the proof of Claim 3.2, we have

$$2 = \chi(S) \leqslant v - e + d \leqslant \frac{e}{3} - e + \frac{2e}{3} = 0,$$

which is absurd. □

Fix then a vertex x of $\overline{\Gamma}_1$ of valency $k \leqslant 5$.

Claim 3.6. *There is a family of n_2 parallel edges in Γ_1 which are incident to x and which connect x to a parallel vertex.*

Assuming the truth of this claim for the moment, Lemma 2.4 guarantees that Γ_1 contains an S-cycle, and so n_2 is even by Lemma 2.2(1) and Remark 1.2. Hence $n_2 \geqslant 4$. On the other hand, the claim also implies that $n_2 \leqslant 4$, as Lemma 2.11 forces the inequality $n_2 \leqslant n_2/2 + 2$ when $n_2 \geqslant 3$. Therefore the proof of Theorem 0.1 when $n_2 \geqslant 3$ will be reduced to the consideration of the case where $n_2 = 4$.

Proof. The ends of edges in Γ_1 incident to x, and which connect x to a parallel vertex can be divided into $k \leqslant 5$ families, each belonging to mutually parallel edges of Γ_1. If every end at x belongs to an edge of Γ_1 connecting x to a parallel vertex, then we have at least $(\Delta n_2)/5 \geqslant n_2$ successive ends at x belonging to mutually parallel edges of Γ_1, completing the proof of the claim.

Assume then that there is at least one end incident to x which belongs to an edge of Γ_1 connecting x to an antiparallel vertex. By Claim 3.4, there are at least $(\Delta - 1)n_2$ ends at x which belong to edges of Γ_1 connecting x to parallel vertices. Then at x, there is a family of at least $((\Delta - 1)n_2)/4 \geqslant n_2$ successive ends which belong to mutually parallel edges of Γ_1 connecting y to a parallel vertex. □

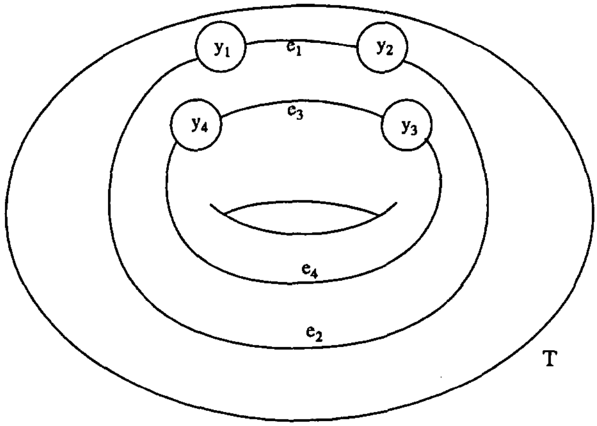

Fig. 2.

As noted above, Claim 3.6 reduces us to the case $n_2 = 4$. Assume this and let y_i, $i = 1, 2, 3, 4$, be the four vertices of Γ_2. By Remark 1.2, we may assume that y_1, y_3 are assigned a "+" sign while y_2, y_4 are assigned a "−" sign. We know from Claim 3.6 that incident to the vertex x of Γ_1, there are $n_2 = 4$ successive ends which belong to mutually parallel edges of Γ_1 connecting x to a parallel vertex (perhaps x itself). Let e_i, $i = 1, 2, 3, 4$, be four such edges. Using the parity rule, one can easily see that either the four edges form an extended S-cycle in Γ_1 or they form two S-cycles with disjoint label pairs in Γ_1. The former case is impossible by Lemma 2.11. In the latter case we may assume that $\{e_1, e_2\}$ form one S-cycle with label pair $\{1, 2\}$ and $\{e_3, e_4\}$ form the other S-cycle with label pair $\{3, 4\}$. By Lemma 2.8, the two cycles $e_1 \cup e_2 \cup y_1 \cup y_2$ and $e_3 \cup e_4 \cup y_3 \cup y_4$ in Γ_2 are essential and disjoint (see Fig. 2).

Now let us return to the reduced graph $\overline{\Gamma}_2$. We may consider Fig. 2 to be a subgraph of $\overline{\Gamma}_2$. The cycles $e_1 \cup e_2 \cup y_1 \cup y_2$ and $e_3 \cup e_4 \cup y_3 \cup y_4$ divide T into two annuli, and since each vertex of $\overline{\Gamma}_2$ has valency 6 (Claim 3.2), it may be argued that in each of these annuli, there is an edge of $\overline{\Gamma}_2$ connecting y_1 and y_3. But then in $\overline{\Gamma}_2$, there are distinct edges which connect y_1 and y_3. As y_1, y_3 are parallel vertices, we obtain a contradiction to Claim 3.3. This completes the proof of Theorem 0.1 when $n_2 \geqslant 3$.

4. Proof of Theorem 0.1 when $n_2 \leqslant 2$

In this section we complete the proof of Theorem 0.1 by dealing with the special cases $n_2 = 1$ and $n_2 = 2$. In fact we obtain sharp estimates for $\Delta = \Delta(r_1, r_2)$ and use these to prove Theorem 0.2. First we show

Lemma 4.1. *Suppose that $n_2 = 1$, then $\Delta = 1$.*

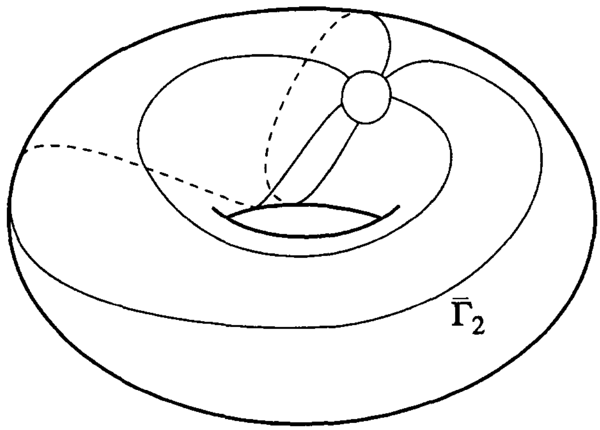

Fig. 3.

Proof. Suppose to the contrary that $\Delta \geqslant 2$. Recall that there are no trivial loops in Γ_2, and so the same holds for $\overline{\Gamma}_2$. On the other hand, our hypothesis that $n_2 = 1$ implies that $\overline{\Gamma}_2$ has only one vertex, and so every edge of $\overline{\Gamma}_2$ is necessarily an essential cycle in T. One may now argue that $\overline{\Gamma}_2$ is a subgraph of the graph illustrated in Fig. 3 (for a proof see [4, Lemma 5.1]). Hence the valency of the vertex of $\overline{\Gamma}_2$ is either 2 or 4 or 6. Correspondingly in Γ_2, we have either $k = 1, 2$ or 3 families of mutually parallel edges. Order these families in a clockwise fashion around the vertex and let p_i be the number of edges in the ith family. Then counting the ends at the vertex we have

$$\Delta n_1 = 2\sum_{i=1}^{k} p_i. \tag{1}$$

On the other hand, Lemma 2.5(3) implies the following inequalities,

$$2n_1 \leqslant \Delta n_1 \leqslant 2k\left(\frac{n_1}{2} + 1\right) = k(n_1 + 2) \leqslant 3(n_1 + 2) \tag{2}$$

with

$$\Delta n_1 = 2k\left(\frac{n_1}{2} + 1\right) \text{ if and only if } p_i = \left(\frac{n_1}{2} + 1\right) \text{ for } i = 1, 2, \ldots, k. \tag{3}$$

As $n_1 \geqslant 3$ (Lemma 2.3), we immediately deduce that $k > 1$ and that $\Delta \leqslant 5$. Furthermore, if $\Delta = 5$, then the second and third inequalities in (2) are equalities, which forces the identity $n_1 = 3$ and so by (3), $p_i = 2.5$ for each i, which is clearly false. Thus we must have $\Delta \leqslant 4$.

Claim. $\Delta = 3$.

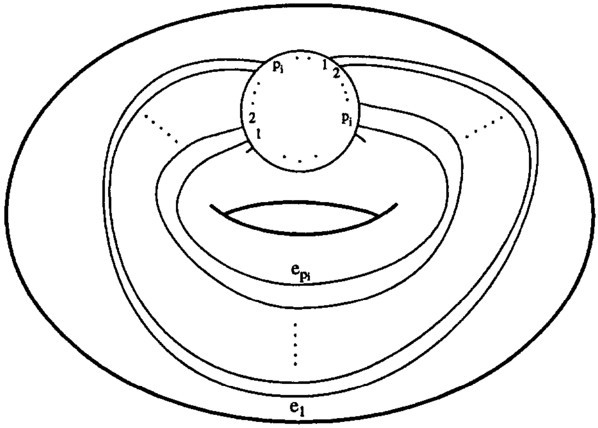

Fig. 4.

Proof of the claim. If $\Delta \neq 3$, then it must be either 2 or 4. Assume this and rewrite identity (1) as

$$\sum_{i=1}^{k} p_i = \left(\frac{\Delta}{2}\right) n_1.$$

Then if $e_1, \ldots, e_{p_i}$ is one of the families of mutually parallel edges in Γ_2, the eveness of Δ implies that the labels of the ends of these edges are labeled (mod n_1) as shown in Fig. 4. But then by Lemma 2.5(4), we have $p_i = 2$, and the edges form an S-cycle. Thus for each $i = 1, \ldots, k$ the ith family of edges forms an S-cycle. Finally, as $n_1 \geqslant 3$, the first two S-cycles have distinct labeling pairs, which contradicts Lemma 2.5(1). The claim is therefore proved.

Assume then that $\Delta = 3$. Substituting this value into identity (1) shows that n_1 is even, and so by Lemma 2.3 we have $n_1 \geqslant 4$. On the other hand, if we suppose now that $k = 2$, inequalities (2) show that $n_1 \leqslant 4$, and so $n_1 = 4$. Note then that $\Delta n_1 = 2k(n_1/2 + 1)$, and therefore from identity (3) we obtain $p_1 = p_2 = 3$. Appealing to Fig. 3 shows that there is an edge of Γ_2 having identical labels at both of its ends, which contradicts the parity rule. Thus we must have $k = 3$.

We observed in the previous paragraph that $n_1 \geqslant 4$ is even, say $n_1 = 2m$ where $m > 1$. Fix $i \in \{1, 2, 3\}$ so that $p_i = \max\{p_1, p_2, p_3\}$, and let $e_1, \ldots, e_{p_i}$ be the associated family of parallel edges. Then from identity (1) we see that $p_i \geqslant m$. But by Lemma 2.5(3), $p_i \leqslant m + 1$, and therefore $p_i = m$ or $p_i = m + 1$. The latter is impossible, for if it held, identity (1) would imply that the ends of e_{p_i} (see Fig. 5) would be labeled $p_i = m + 1$ and $p_1 + p_2 + p_3 + 1 = 3m + 1 \equiv m + 1 \pmod{n_1}$, which contradicts the parity rule. Thus $p_i = m$. Given our choice of p_i and considering identity (1), we see that $p_1 = p_2 = p_3 = m$. Therefore Γ_2 is a graph shown in Fig. 6.

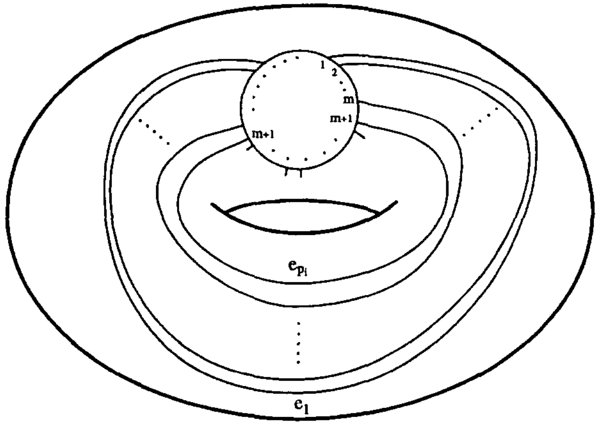

Fig. 5.

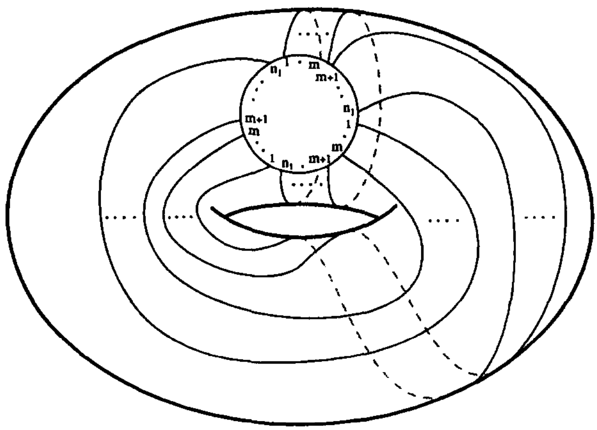

Fig. 6.

From Fig. 6, we see that in Γ_2 there are two Scharlemann cycles of length 3, one with label pair $\{m, m+1\}$ and the other with the different label pair $\{1, n_1\}$. But by the minimality assumption on n_1, there can never be two such Scharlemann cycles. This is proved using an argument similar to that found in the proof of [13, Lemma 2.2]. This final contradiction completes the proof of Lemma 4.1. □

Theorem 4.2. *Genus one knots in S^3 satisfy the cabling conjecture.*

Proof. Let $K \subset S^3$ be a genus one knot which is not a cabled knot, but which admits a surgery slope which yields a reducible manifold. According to [12], K is not a satellite knot. Hence K is a hyperbolic knot, i.e., $M = S^3 - \operatorname{int} N(K)$ is a hyperbolic manifold. According to [3], $M(0)$ is an irreducible manifold. Thus Lemma 2.1 shows that the

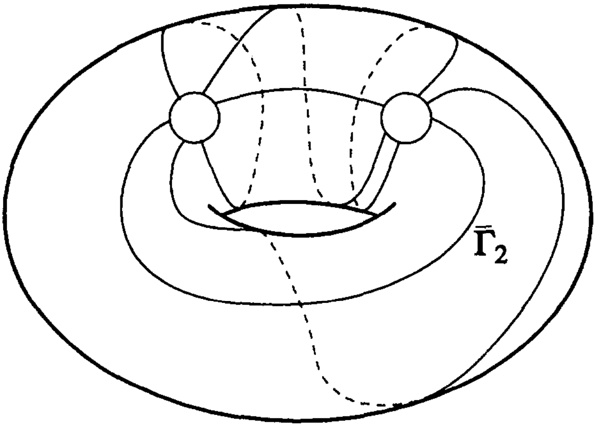

Fig. 7.

hypothesized genus one Seifert surface for K will produce an essential nonseparating torus T in $M(0)$.

Now suppose that $M(r)$ is a reducible manifold for some slope r. By [5,6], r is an integral slope with $|r| \geqslant 2$. But then $\Delta(0,r) = |r| \geqslant 2$, which contradicts Lemma 4.1. Thus there is no such knot, and the proof is complete. □

Lemma 4.3. *Suppose that $n_2 = 2$, then $\Delta \leqslant 3$.*

Proof. We assume that $\Delta \geqslant 4$ so as to obtain a contradiction. It may be argued that the reduced graph $\overline{\Gamma}_2$ of Γ_2 in T, is a subgraph of the graph illustrated in Fig. 7 (for a proof see [4, Lemma 5.2]). Hence there are at most five families of mutually parallel edges in Γ_2 incident to each of the two vertices, and one of the five families is a set of parallel loops. Let $p_i \geqslant 0$ be the number of edges in each of the five families, $i = 1,2,3,4,5$, with p_1 being the number of the parallel loops. Note then that

$$\Delta n_1 = 2p_1 + p_2 + p_3 + p_4 + p_5. \tag{4}$$

Now by Lemma 2.5(3), $p_1 \leqslant (n_1/2 + 1)$, and by Lemma 2.6 $p_i \leqslant (n_1 - 1)$ for $2 \leqslant i \leqslant 5$. Thus identity (4) shows that $\Delta \leqslant 4$, and so we may assume that $\Delta = 4$. Substituting this value for Δ into (4) and using a similar reasoning, we see that $p_i > 0$ for each $i = 1, \ldots, 5$. So we may assume that Γ_2 is as shown in Fig. 8. It follows that without loss of generality, we may assume that $p_1 + p_2 + p_3 \geqslant 2n_1$.

Now consider the labels around one of the two vertices $\{y_1, y_2\}$ of Γ_2, say around y_1, and let $e_1, \ldots, e_{p_1}$ be the edges of Γ_2 which form the p_1 parallel loops. Let $r \in \{1, 2, \ldots, n_1\}$ be the label of the other end of e_{p_1}. Now from the form of Γ_2 (see Fig. 8), $r \equiv p_1 + p_2 + p_3 + 1 \pmod{n_1}$. On the other hand, by construction

$$2n_1 < p_1 + p_2 + p_3 + 1 \leqslant \left(\frac{n_1}{2} + 1\right) + 2(n_1 - 1) + 1 = \frac{n_1}{2} + 2n_1 < 3n_1.$$

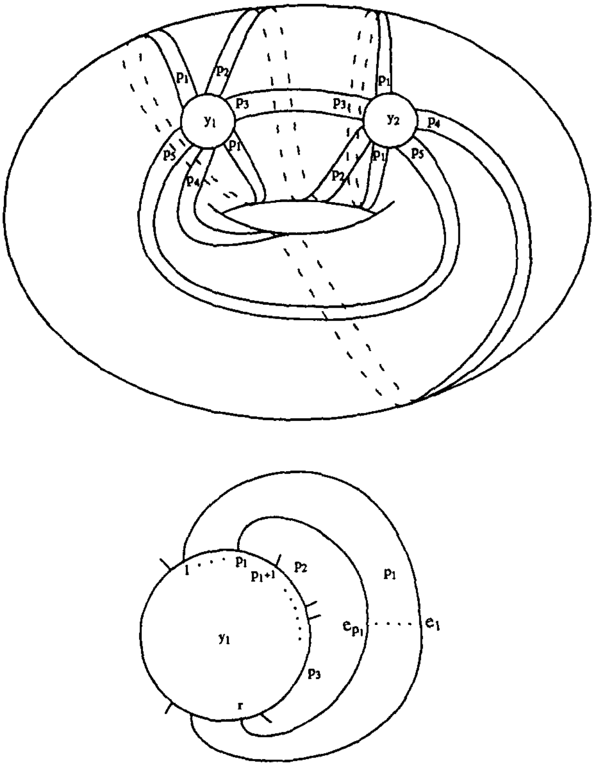

Fig. 8.

Hence $p_1 + p_2 + p_3 + 1 = r + 2n_1$. Then

$$p_1 = r + 2n_1 - (p_2 + p_3 + 1) \geqslant r + 2n_1 - 2(n_1 - 1) - 1 = r + 1,$$

i.e., $1 \leqslant r \leqslant p_1 - 1$. Therefore there are two edges amongst $\{e_1, \dots, e_{p_1}\}$ which have p_1 as a common label. Applying Lemma 2.5(4) we see that $\{e_{p_1-1}, e_{p_1}\}$ are part of an S-cycle in Γ_2. In particular $r = p_1 - 1$. But we have already seen that $p_1 + p_2 + p_3 + 1 = r + 2n_1$, and so $p_2 + p_3 = 2n_1 - 2$. From Lemma 2.6 we deduce that $p_2 = p_3 = n_1 - 1$.

Next apply the same argument to the family of edges forming a loop at y_2 to deduce that they also contain an S-cycle in Γ_2 which is situated as indicated in Fig. 9, and also has label pair $\{p_1 - 1, p_1\}$ by Lemma 2.5(1).

Suppose now that the two vertices of Γ_2 are parallel. According to Lemma 2.3 and Lemma 2.5(3), $3 \leqslant n_1 \leqslant 4$. On the other hand, Γ_2 contains an S-cycle, so n_1 is even by Lemma 2.2(2) and Remark 1.2. Hence $n_1 = 4$, $p_2 = p_3 = n_1 - 1 = 3$, and

$$2 \leqslant p_1 \leqslant \frac{n_1}{2} + 1 = 3.$$

In the case $p_1 = 2$, all S-cycles in Γ_2 are labeled $\{1, 2\}$. The labels at the ends of the family of $p_2 = 3$ parallel edges, connecting the two vertices in Γ_2, are therefore

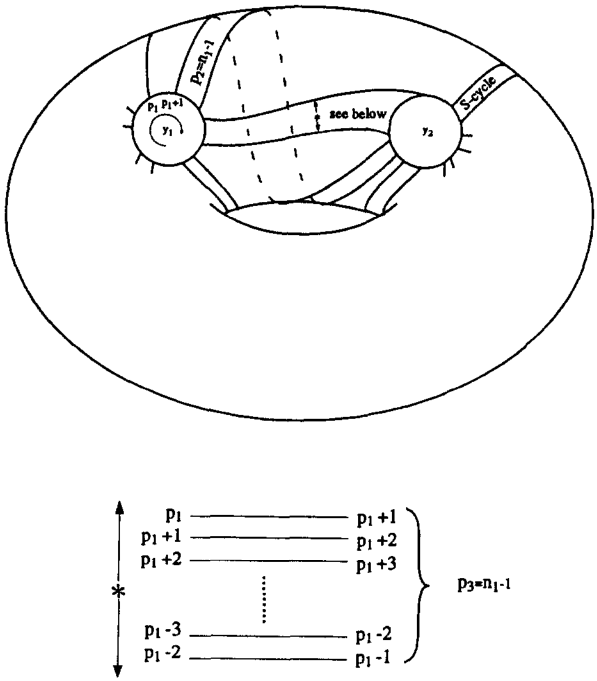

Fig. 9.

determined and it can be seen that there is an edge both of whose labels are 4, which contradicts the parity rule. A similar argument shows that it is impossible for p_1 to be 3. Thus y_1 and y_2 cannot be parallel.

Assume then that the two vertices of Γ_2 are antiparallel. We may fill in the labels at y_2 and then apply the parity rule to the ends of the p_2 edges from the second family to deduce that all the vertices of Γ_1 are parallel (see Fig. 9). But this is impossible as the existence of a loop in Γ_2 means that there are two antiparallel vertices in Γ_1. This contradiction completes the proof of Lemma 4.5. □

References

[1] M. Culler, C. Gordon, J. Luecke and P. Shalen, Dehn surgery on knots, Ann. of Math. 125 (1987) 237–300.

[2] M. Eudave-Muñoz, Band sum of links which yield composite links: The cabling conjecture for strongly invertible knots, Trans. Amer. Math. Soc. 330 (1992) 463–501.

[3] D. Gabai, Foliations and the topology of 3-manifolds III, J. Differential Geom. 26 (1987) 479–536.

[4] C. Gordon, Boundary slopes of punctured tori in 3-manifolds, Preprint.

[5] C. Gordon and J. Luecke, Reducible manifolds and Dehn surgery, Topology, to appear.

[6] C. Gordon and J. Luecke, Only integral Dehn surgery can yield reducible manifolds, Math. Proc. Cambridge Philos. Soc. 102 (1987) 97–101.

[7] C. Gordon and J. Luecke, Knots are determined by their complements, J. Amer. Math. Soc. 2 (1989) 371–415.
[8] C. Gordon and R. Litherland, Incompressible planar surfaces in 3-manifolds, Topology Appl. 18 (1984) 121–144.
[9] E. Luft and X. Zhang, Symmetric knots and the cabling conjecture, Math. Ann., to appear.
[10] W. Menasco and M.B. Thistlethwaite, Surfaces with boundary in alternating knot exteriors, J. Reine Angew. Math. 426 (1992) 47–65.
[11] D. Rolfsen, Knots and Links (Publish or Perish, Berkeley, CA, 2nd ed., 1990).
[12] M. Scharlemann, Producing reducible 3-manifolds by surgery on a knot, Topology 29 (1990) 481–500.
[13] Y. Wu, The reducibility of surgered 3-manifolds, Topology Appl. 43 (1992) 213–218.